More on Quantum Measurement: Of lunch-boxes and school-kids

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The most famous example of two non-commuting observables are position and momentum. The properties of these operators are a little complicated because their spectra are continuous. It is easier to consider the case of measuring angular momentum or even better the spin of an $S=1/2$ system. The three Cartesian components of $\hat{S}$ do not commute and we have the commutation relations $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$. However we can very well measure any of these individual quantities and we can also perform a sequence of measurements and analyse the results. In the absence of magnetic interactions in the Hamiltonian the resulting state vectors after the measurement are independent of time, which is another simplification. In fact to discuss the results of quantum mechanics let us not use any mathematics at all. Below I will discuss a ‘real life’ example, involving little or no mathematics.

As our ensemble we take a class of schoolkids. Each of these kids has a lunchpacket that consists of three items. They all have a turkey or roastbeef sandwich ($t$ or $r$), a coke or a sprite to drink ($c$ or $s$) and an apple or an orange ($a$ or $o$) for dessert. Our measurement consists of asking a kid what is in the lunchbag, and getting statistics on the ensemble (the class). However, we can ask only one question at a time. For example: "everybody with a turkey sandwich stand to the right". But not: "All that have an orange and a coke please stand on the left". That is asking two questions at once, and in the analogy with the spin system reflects the impossibility to simultaneously measure non-commuting observables. In fact any ‘measurement’ we do should obey the laws of quantum mechanics. Our goal is to characterize the distribution of lunchbags (e.g how many $tca, rca, tsa, rco, ...$ etc, are there. Can we do this? If things behaved classically, easily. But not in the quantum world. Let us try. We would first ask all kids who has turkey and who has roast beef, and partition them into two groups. Then we would ask the turkey
group who has a coke and set them apart. Fine. We already have an ensemble that has both a turkey and a coke, right? Let us check, and ask again. Who has a coke? Everybody has a coke. Now, who has a turkey sandwich? Oops. This doesn't work. Only about half of them has turkey. Asking the coke question destroyed the information we had on the turkey. In the quantum world it is impossible to isolate a group where everybody has both a coke and turkey. Asking the question changes the ensemble. This is fairly easy to understand mathematically, describing an ensemble as a vector in Hilbert space, that rotates under measurement, but it certainly does not make much sense when asking about lunch bags.

The above is a representation in as simple a language as possible of some puzzling properties of quantum mechanics. The essence is that according to quantum mechanics (sometimes) we cannot create an ensemble that for sure will yield definite values for two non-commuting observables. This is the content of the Heisenberg uncertainty principle. The precise formulation would be

$$\Delta A \Delta B \geq \frac{1}{2} |\{\hat{A}, \hat{B}\}|.$$

For a proof and discussion see Cohen-Tannoudji pages 286-289.

It is often stated as "one cannot measure the precise value of \(\hat{A}\) and \(\hat{B}\) simultaneously". This is a very incomplete statement of the principle and it has led to all kinds of ingenious constructions to violate the principle. It is much easier and complete to interpret the principle in a different way. There is no problem to measure \(\hat{A}\) or \(\hat{B}\), and for each measurement (either \(\hat{A}\) or \(\hat{B}\), but not both) on an individual system you get definite results. However, for certain pairs of eigenvalues of \(\hat{A}\) and \(\hat{B}\), \((a_i, b_j)\) say, it is in principle impossible (according to QM) to prepare an ensemble such that all of the measurements on this ensemble yield precisely the result \(a_i\) if you measure \(\hat{A}\) and \(b_j\) if you would measure \(\hat{B}\). In contrast there is no problem in preparing an ensemble such that every member would yield \(a_i\) if you measure \(\hat{A}\). You might put in some effort to appreciate the precise translation of the mathematical formulation of the uncertainty principle into words. It is a little easier if the commutator \([\hat{A}, \hat{B}]\) is a constant, since then
no ensemble will yield the same value for $A$ and $B$ for all elements in the ensemble. So necessarily there is a spread, and the minimum spread depends on the commutator. In the general formulation the minimum spread depends on an expectation value and hence on the state under consideration. Note that quantum mechanics actually does not preclude that individual systems have definite values for all observables. It does say that within the realm of quantum mechanics you cannot create an ensemble to prove it. Also note that it is impossible to discuss the uncertainty principle using a single system. It is perfectly possible to have an experiment where you measure $\hat{A}$ then $\hat{B}$ then $\hat{A}$ then $\hat{B}$ and find nothing weird: measurement of $\hat{A}$ yields $a_i$ twice, while the measurement of $\hat{B}$ yields $b_j$ in both cases. This is quite a possible outcome of this experiment. But beforehand you cannot be certain that it will happen that way. It is impossible to create an ensemble where all elements necessarily behave in this fashion. Of course you might be lucky and by chance, using small enough ensembles one can easily violate the Heisenberg uncertainty principle. That is all part of statistics.

Let us discuss another hair raising situation. A long standing controversy is the so-called Einstein-Podolski-Rosen Paradox (EPR). EPR sought for the properties of individual systems obeying the laws of quantum mechanics. In essence all parties can agree on the fact that a measurement can change the system. So in the example above if I ask Mary if she has a coke, afterwards she might no longer have the roastbeef sandwich that she started out with. However, the issue at hand is different. EPR thought it would be possible that each lunchbag has a definite content before measurement, and we are simply looking what is in it. By looking at one piece of information we might, in the convoluted act of measurement, change another piece of information in ways that are hard to predict. This would then be the reason that one cannot prepare well specified ensembles, which are themselves prepared by measurements. It might be that we simply have too little control over the act of measurement (at present ?). Quantum afficionados tend to think differently about what happens during a measurement on an individual system. Their idea is that by measuring you force the microsystem to take a position. It is like flipping a coin at the moment of measurement. "Choose my dear electron! Up or Down?" By the act of measurement you force the system into an eigenstate of the corresponding observable,
and it does so with probabilities predicted by quantum mechanics. The precise outcome of an individual experiment is unpredictable in principle. If one reads initial accounts of the Heisenberg uncertainty principle, they very much reflect the viewpoint of EPR. Heisenberg himself for example discusses how measuring position necessarily changes the momentum of an electron. The later accepted viewpoint according to the so-called Kopenhagen interpretation is rather convoluted in that they use classical mechanics to describe the measuring apparatus and so there is a mysterious connection between the quantum and classical system. However, I think that the above stated position of the quantum afficionado reflects the attitude of many scientists in the field. It was my position until I wrote these notes.

Let us adjust our lunch bag parable a little so that we can describe the EPR line of thought in trivial terms. What if we could gain information about what is in a lunchbag without asking a question? Let us set up the experiment in a tricky way. Say we know that the lunchbags are handed out in complementary pairs. Each pair contains both turkey and roastbeef, an apple and an orange, a coke and a sprite. So a pair of lunchbags might consist of $tca & rso$ or $rso & tca$ and so forth. We look when the lunchbags are handed out and keep track of the corresponding pairing of the kids. The actual quantum experiment consists for example of two spin $1/2$ atoms in an overall $S=0$ state. You will discuss it yourself later on, working through a set of questions... Back to the kids. Let's say, Lois and Clark form a pair. Now we take Clark out on the playground and ask him about his sandwich. "Turkey he says. I would like salami!" Lois doesn't even know we asked, but we now know that she has roastbeef without asking her (or perturbing her lunchbag). If we would ask her she would say roastbeef 100% of the time. However, we don't need to ask her about her sandwich as we know already. Instead we ask Lois about her drink. "I have a coke she says". After we ask the coke question she might no longer have roastbeef, but if we assume she has something definite in her lunchbag, before the coke question it was most definitely roastbeef and a coke. So this is a smart measurement that shows it makes perfect sense that every lunchbag has something definite in it and by measuring we simply find out what it is. Only, by asking one specific question we might change the content of the lunchbag in other respects, and in unpredictable ways. At the
time EPR wrote their paper this interpretation was in no conflict with any piece of data whatsoever. It was just an interpretation that should have appealed as something far more rational than flipping a coin at the time of measurement. If we take the alternative quantum interpretation about what actually happens, the EPR experiment is seen to take on all of its weirdness. Asking Clark what is in his lunchpacket forces him to take a position. Clark flips a coin to make a decision. "Turkey". If we now would ask Lois about her sandwich she will say roastbeef for sure. So she flips her coin too, but it always yields the same result. If we wouldn't have asked Clark it would give a fifty-fifty result, but now it yields a 100%. Now, Lois nor her coin knows anything about our asking Clark. To put it in the extreme: flipping a coin in Tokyo determines the outcome of the flipping of the coin in New York. That doesn't make sense. The EPR interpretation is far more reasonable: if we assume there is something definite in the lunch bag, there is nothing strange about us knowing what is in Lois's lunch bag if we know what Clark has, given they form a perfect pair.

However, EPR did something more. They claimed that physical theories should describe ‘reality’, which means that quantum mechanics should allow for ensembles of completely specified lunchbags. This it did not, and therefore the theory was not quite up to par. Quantum theory was incomplete. In order to describe ensembles of well specified $\hat{S}_x, \hat{S}_y, \hat{S}_z$ the structure of the theory needs to be changed completely. If we use the concept of Hilbert space, operators and eigenvalues it can not accommodate EPR's reality. Quantum theory was too successful to discard it, just because of a difference in interpretation that had apparently no measurable consequences. Glad we didn't. In my opinion the more reasonable thing to do would have been to adopt the EPR interpretation but live with the fact that quantum mechanics only describes ensembles that can actually be prepared by measurements. Measurements unfortunately tend to perturb the system such that no fully specified ensembles can be prepared. Or perhaps they could and one might make further advances, necessarily leading to a new theory. In essence this would mean to say EPR might be right, but quantum mechanics seemed to do the job in practice. It would have kept the search for alternative theories alive but they would necessarily have the same statistics as quantum mechanics, which has served us very well.
This was the situation until John Bell came around in the 1960’s. He showed that the EPR interpretation might lead to different results from the usual quantum theory for some experiments. And he used the EPR experiment to show it. This is how it works in terms of lunchbags. If EPR's position is right then in fact I can construct what was in Lois's lunchpacket from the pairing experiment. From Clark's answer I know she had roastbeef, and by our question we also know she has a coke. We are simply assuming that the question to Clark could not possibly have affected Lois's lunch box. There is no unpredictable act of measurement that has a range from New York to Tokyo. Let us assume therefore for the sake of argument that EPR are right. Every lunchbox has a definite content and by doing the pairing experiment I can determine two items in a lunchbox. Now we take our whole class and do three types of experiment starting from identical ensembles in each experiment. In the first experiment we use the pairing experiment to determine if somebody has a turkey sandwich and a coke. By assumption she would then have either an orange or an apple as the third item. If we do this for the whole first ensemble we can write

\[ n[t,c] = n[t,c,o] + n[t,c,a] \]

where \( n[t,c] \) denotes the number of kids in the ensemble that have both a turkey and a coke, and so forth. In the next group we determine the number that has a sprite and an orange, in the third group turkey and orange. In total we would then have the following relations, assuming the minimal EPR conditions

\[ n[t,c] = n[t,c,o] + n[t,c,a] \]
\[ n[s,o] = n[t,s,o] + n[r,s,o] \]
\[ n[t,o] = n[t,c,o] + n[t,s,o] \]

From this we can derive the so-called Bell inequality:

\[ n[t,c] + n[s,o] \geq n[t,c,o] + n[t,s,o] = n[t,o] \]

This is an inequality that one can test in an actual experiment, as one can make a spin zero pair, let it fly apart and measure the spin in different directions for the particle in Tokyo and the particle in New York. We will discuss the full details of the precise quantum treatment later on. But the outcome is that the usual treatment of quantum mechanics is in conflict with the above analysis based on the assumptions of EPR.
Quantum mechanics violates Bell's inequalities. At the time of the EPR paper (1935) people couldn't really say if EPR was right or the standard Copenhagen interpretation was right. Neither did contradict any experiment. It appeared simply a matter of interpretation. With Bell however, there was a testable hypothesis. Experiments were done in the seventies, and the experiments by Alain Aspect are perhaps best known (though not the first). The technical details and fine print are rather involved, but the conclusion was that the traditional laws of quantum mechanics are correct. So one cannot assume that individual particles actually have definite values for $\hat{S}_x, \hat{S}_y, \hat{S}_z$ and we are simply determining what they are, although perturbing these values in the process.

Does this mean that we have to accept the alternative interpretation? Flipping a coin in Tokyo determines the outcome of the flipping of a coin in New York? Not in my opinion. This 'making a choice during the measurement' aspect appears to be an act of human imagination. Us trying to understand what we cannot grasp. I think it is better to take a very mundane position. Quantum mechanics describes the statistical outcomes of complete experiments. In doing the measurement in Tokyo I am preparing a specific ensemble. The subsequent measurement in New York is described using this new ensemble. From the perspective of quantum mechanics the pairing experiment is no different from first asking Lois if she has roastbeef and then asking if she has a coke. A measurement is a measurement, and if a measurement in Tokyo tells you something about the situation in New York, you have to adjust the ensemble accordingly. Of course this is nothing more than using the laws of quantum mechanics which are very definite for this type of experiment. What is hard to understand is how there can be such a strong correlation between two distant particles, which cannot be assumed to individually have definite properties, while as a pair they do. Quantum mechanics gives us the mathematical prescription but it does go against common sense, and this is illustrated very vividly by the flipping the coin at the time of measurement picture. This phenomenon is called entanglement in the literature.